Simultaneous state and parameter estimation for dynamical systems using Observers

Vishal Vasan & Vishal Neeraje *

Dynamical systems serve as models to a wide array of applications in the natural sciences and engineering. A major challenges in using such models for prediction is knowing the value of parameters or coefficients of the model. Since parameters in a dynamical system can change both qualitative and quantitative aspects of a solution, their accurate estimation is of utmost importance. The typical recourse to estimate parameters is to obtain data from measurements of the state-vector, however we can only measure a function of the state and not the full state-vector. In our recent work, we investigate how one can estimate parameters for dynamical systems from partial observations of the state. Our goal is to simultaneously recover the full state as well as the parameters.

Consider a dynamical system

$$\frac{dy}{dt} = A_\theta y + f(y)$$

where $y(t) \in \mathbb{R}^n$, A_θ is an $n \times n$ matrix that depends on parameters $\theta \in \mathbb{R}^p$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a function that represents nonlinear effects in the model. Furthermore, we assume we have access to observations $z(t) = b^T y(t)$ where $b \in \mathbb{R}^n$ is a constant vector. Note the observation is lower dimensional than the state-vector. A popular method to recover $y(t)$ from the observations $z(t)$ is due to [1] given by defining the 'observer'

$$\frac{d\tilde{y}}{dt} = A_{\tilde{\theta}}\tilde{y} + f(\tilde{y}) + l(b^T\tilde{y} - z)$$

The vector $\tilde{y}(t) \in \mathbb{R}^n$ represents the current estimate of the state-vector. The observer is simply the original model with a feedback term representing the current observation error. In the traditional implementation of an observer, that is when $\tilde{\theta} = \theta$, one chooses $l \in \mathbb{R}^n$ such that $\|\tilde{y}(t) - y(t)\| \to 0$ as $t \to 0$ irrespective of the initial conditions $\tilde{y}(0), y(0)$. In the case of interest to us, since the parameters of the model are unknown, $\tilde{\theta} \neq \theta$ and hence one does not expect to recover the state eventually.

Assuming that the only source of error in the observation is due to the parameter mismatch, we propose to estimate the parameter by minimising the long-time observation error, *i.e.*

$$\theta^* = \arg\min \frac{1}{T_0} \int_{T_0}^{T+T_0} (b^T\tilde{y}(t;\tilde{\theta}) - z(t))^2 \, dt$$

where T, T_0 are positive real numbers and $T_0 >> 1$ is to ensure that the transient error due to the incorrect initial condition for $\tilde{y}$ is sufficiently small.

In the talk, I'll present numerical evidence that the above minimisation problem can indeed recover the parameter and the state accurately. The models considered will include both linear and nonlinear differential equations. In the linear case, we provide a convergence result and sufficient conditions on the kinds of models for which we can expect simultaneous state and parameter estimation. Models of particular interest to us include Lorenz-63 [2], the rotating shallow-water equations and a model for water-waves over variable topography [3]. The algorithm can be readily adapted to the case including noisy observations and I will discuss the relative advantages of the proposed approach compared to similar methods and the main computational challenges in the present method.

References

[1] David G. Luenberger, *Introduction to Dynamic Systems: Theory, Models, and Applications*, Wiley, 1979.

[2] E. N. Lorenz. Deterministic nonperiodic flow. *Journal of Atmospheric Sciences*, 20.2:130–141, 1963.

[3] V. Vasan and Manisha and D. Auroux. Ocean-depth measurement using shallow-water wave models. *Studies in Applied Mathematics*, 147.4: 1481–1518, 2021.

*V. Vasan is with the International Centre for Theoretical Sciences email: vishal.vasan@icts.res.in V. Neeraje is with the International Centre for Theoretical Sciences, email: vishal.neeraje@icts.res.in